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Swash overtopping a truncated plane beach

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Swash on a plane beach is modelled by using a solution of the shallow-water equations due to Shen & Meyer (1963). The equations are used in a form appropriate for a plane at a finite angle to the horizontal. The beach is cut-off at a level below that of the maximum run-up, and the water is taken to fall freely over the end of the beach. An explicit solution is found which permits evaluation of the overtopping flow and total volume for one swash event.

1. Introduction

In coastal protection, whether against flooding or to prevent wave disturbance in harbours, it is often valuable to be able to estimate the flow of water over the top of a structure, or natural feature. When the mean water level is below the crest of such a feature any overtopping is due to the run-up from incident waves. Here we consider a sloping structure so that run-up above the mean water level has the form of a relatively thin swash. We also simplify the crest of the structure by assuming it has a sharp edge over which water falls when it reaches the top. We construct an analytical solution that we believe is the first of its type.

Previous work has until recently been almost entirely based on measurements of laboratory experiments (Franco, De Gerloni & van der Meer 1994; van der Meer & Janssen 1995; Besley, Stewart & Allsop 1998). Some numerical studies, have also been made (Jervis & Peregrine 1996; Dodd 1998; Hu, Mingham & Causon 2000).

Water wave motion at a shoreline on a sloping structure takes two different forms. If there is no breaking of the waves the motion is smooth as the incident waves are reflected and the wave motion can be described well either by linear theory for slopes of $O(1)$, see Whitham (1979) for a full account, or by Carrier & Greenspan's (1958) solutions for the nonlinear shallow-water equations.

On the other hand, if waves break near the shoreline, which is almost always the case for beaches of gentle slope and often occurs on steeper slopes, then flow in the swash zone, is quite different. Meyer & Taylor (1972) discuss the boundary between these two types of flow in the context of shallow-water theory. In the breaking case, for each wave crest reaching the shoreline there is a new 'swash event' generated. This runs up the slope until overtaken by another such event, or until it drains back under gravity.

The only theoretical description of such an event is in the context of shallow-water theory, where waves have broken and formed a bore which then meets the shoreline. The swash from a 'uniform bore' on a plane beach, travelling over still water before meeting the shoreline is described by Shen & Meyer (1963). In shallow-water theory bores are represented by a moving discontinuity in water depth and velocity. Ho & Meyer (1962) show that as the bore meets the shoreline its height 'collapses' to zero. In the context of shallow-water theory this singularity emits a swash event, singular at the instant of creation, but characterized by an initial velocity which we replace

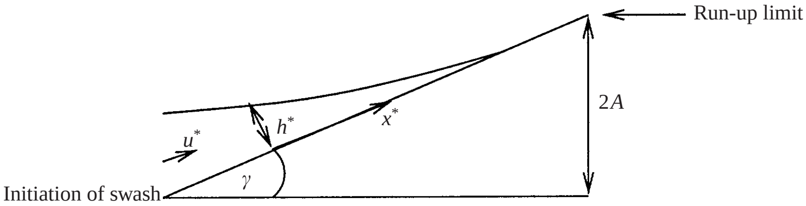

FIGURE 1. Definition sketch.

with an equivalent parameter, the height of run-up above the initiation point of the swash event.

The shallow-water equations and their characteristics are presented in § 2. The swash solution of Shen & Meyer (1963) is given in § 3 and its modification due to overtopping of a finite plane beach is in § 4. The concluding discussion of § 5 includes the expression for the amount of water that overtops in each swash event.

2. Shallow-water equations

The nonlinear shallow-water equations are a good approximation to wave motion in the surf and swash zone (Stoker 1957). Normally they are used for beaches of gentle slope. However, since we are only considering the swash zone, where even for steep slopes a breaking wave generates a thin sheet of swash, we write the equations for two-dimensional flow over a beach sloping at an angle γ to the horizontal:

$$h^*_{t^*} + (h^* u^*)_{x^*} = 0, \tag{2.1}$$

$$u^*_{t^*} + u^* u^*_{x^*} + g \cos\gamma \, h^*_{x^*} = -g \sin\gamma, \tag{2.2}$$

where x^* is measured up the slope, t^* is time, $h^*(x^*, t^*)$ is the water thickness and $u^*(x^*, t^*)$ is the water velocity parallel to the beach. The $*$ denotes dimensional variables and g the acceleration due to gravity. These equations may be derived by assuming that water accelerations perpendicular to the slope and the shear in the water are negligible (Peregrine 1972). This last assumption may fail in the backwash, see Peregrine's (1974) description of the waves that may then arise.

The equations can be made dimensionless and free of both the parameters g and γ. We relate lengths to the vertical excursion of the undisturbed swash, $2A$, see figure 1. That is, the swash event has a maximum height of run-up on an unbroken plane beach of $2A$ above its point of initiation, which is the lower boundary of the swash. We place the origin of x^* at this lower boundary, and measure it upslope so that the swash runs up to $x^* = 2A/\sin\gamma$ at the run-up limit. Thus we choose to make x^* dimensionless with $A/\sin\gamma$. The thickness of water h^* is measured perpendicular to the slope, so it is made dimensionless with $A/\cos\gamma$. We use $g\cos\gamma$ and $A/\cos\gamma$ to make t^* and u^* dimensionless.

In order that the scaling for t matches that for x, it needs an extra factor of $\tan\gamma$; then our new scaled dimensionless variables are

$$x = \frac{(\sin\gamma)x^*}{A}, \quad t = (\sin\gamma)t^*\sqrt{\frac{g}{A}}, \quad h = \frac{(\cos\gamma)h^*}{A}, \quad u = \frac{u^*}{\sqrt{gA}}. \tag{2.3}$$

This gives the equations

$$h_t + (hu)_x = 0, \tag{2.4}$$

$$u_t + uu_x + h_x + 1 = 0, \tag{2.5}$$

which are free from any parameters.

To understand the overtopping solution, we need to introduce the local long-wave velocity $c = \sqrt{h}$, and to rearrange the equations into characteristic form

$$\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)\alpha = 0, \tag{2.6}$$

and

$$\left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right)\beta = 0. \tag{2.7}$$

The characteristic variables are

$$\alpha(x,t) = u + 2c + t, \tag{2.8}$$

and

$$\beta(x,t) = u - 2c + t. \tag{2.9}$$

Equations (2.6) and (2.7) imply that α and β are constant along the trajectories (characteristics), C_+, C_-, in (x,t) given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + c \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u - c \tag{2.10}$$

respectively. These are the advancing and receding characteristics and correspond physically to paths of infinitesimal wave disturbances.

3. The swash solution

The effectively discontinuous nature of the swash event generated by bores and breakers is readily seen on most beaches. Shen & Meyer (1963) set out to develop Ho & Meyer's (1962) results for a bore reaching the shoreline. The expectation was that a solution might be found for some further small time interval. Remarkably the paper describes the way in which the whole swash event has much of its flow determined by the initial motion. This is particularly so for the flow close to the instantaneous shoreline which moves up and down the beach under gravity just like a freely moving particle. Another remarkable feature is that the tip of the run-up is very thin, and not just thin because that is necessary for the shallow-water approximation to hold. Waves which do not break, such as Carrier & Greenspan's (1958) solutions, have a moving shoreline with a finite gradient, h_x, for the water depth. However, a swash event caused by a bore is such that $h_x = 0$ at the shoreline and the analytic solution is tangential to the bed. This implies that the actual tip of the solution is invalid in practice because of bed roughness, surface tension or viscosity. Even so, comparisons with numerical solutions for water depths near the tip are remarkably good for water depths greater than 2 mm (Packwood 1980; Titov & Synolakis 1995; Barnes 1996). A numerical version of this type of flow is described in detail in Hibberd & Peregrine (1979).

The swash solution from Shen & Meyer (1963) has a single free parameter, the initial velocity of the shoreline. Here, the scaling with $2A$ effectively fixes that parameter so we need only consider the unique solution with shoreline motion

$$x_s(t) = 2t - \tfrac{1}{2}t^2. \tag{3.1}$$

The depth of water local to the shoreline is

$$h(x,t) = \frac{1}{9t^2}[x_s(t) - x]^2 = \frac{1}{36t^2}(4t - t^2 - 2x)^2. \tag{3.2}$$

Using this form of $h(x,t)$, we find the long-wave velocity, $c(x,t)$, as

$$c(x,t) = \frac{1}{3t}(x_s - x) = \frac{1}{6t}(4t - t^2 - 2x), \tag{3.3}$$

and from (2.4) and (2.5)

$$u(x,t) = \frac{2}{3t}(t - t^2 + x). \tag{3.4}$$

With both $u(x,t)$ and $c(x,t)$ known, the characteristics are found. The advancing characteristics C_+ are

$$x = 2t - \tfrac{1}{2}t^2 - at^{1/3}, \tag{3.5}$$

on which $\alpha = 2$. The receding characteristics C_- are

$$x = 2t - \tfrac{1}{2}t^2 - bt, \tag{3.6}$$

on which $\beta = 2 - \frac{4}{3}b$. The constants a and b are parameters for the family of C_+ and C_- characteristics. The shoreline is common to both families, with $a = b = 0$. Increasing a or b from zero gives the other characteristics. The solution is not useful close to its singularity at the origin of (x,t).

Note that this solution is related to the well-known dam break problem on a horizontal bed, by changing variables to $(\xi, \tau, U, C, H) = (x + \frac{1}{2}t^2, t, u + t, c, h)$, giving the solution

$$U = \frac{2}{3}\left(1 + \frac{\xi}{\tau}\right), \quad C = \frac{1}{3}\left(2 - \frac{\xi}{\tau}\right). \tag{3.7}$$

At $t = \tau = 0$ this is singular, but is usually interpreted as being derived from the initial conditions $H = 1$, $U = 0$ in $\xi < 0$. However strictly all that is needed is that

$$U + 2C = 2 \quad \text{in } \xi < 0. \tag{3.8}$$

Hence other distributions of U and C, or equivalently u and h, satisfying this condition also make sensible initial conditions for this same swash event.

Figure 2 shows both families of characteristics for this solution. Note that strictly only a strip close to the shoreline is fully determined from the initial time and α could be varying with each incoming characteristic. However, in the absence of any other explicit solutions we work with this solution.

In dimensional units the expressions for x_s^*, h^* and u^* are

$$x_s^*(t^*) = 2t^*\sqrt{gA} - \tfrac{1}{2}gt^{*2}\sin\gamma, \tag{3.9}$$

$$h^*(x^*,t^*) = \frac{1}{36gt^{*2}\cos\gamma}(4t^*\sqrt{gA} - gt^{*2}\sin\gamma - 2x^*)^2, \tag{3.10}$$

and

$$u^*(x^*,t^*) = \frac{2}{3t^*}(t^*\sqrt{gA} - gt^{*2}\sin\gamma + x^*). \tag{3.11}$$

For reference in the next section, figure 2 also includes the line in (x,t) where the flow is critical, that is $u = c$. This line is $x = \frac{1}{2}t^2$.

We note that the solution (3.10), (3.11), has a singularity at $t^* = 0$. We can check the consistency of the swash solution with the shallow-water approximation by evaluating the acceleration of water particles perpendicular to the beach. This is more appropriate than simply requiring $h^* \ll A$. For the dimensionless solution a particle

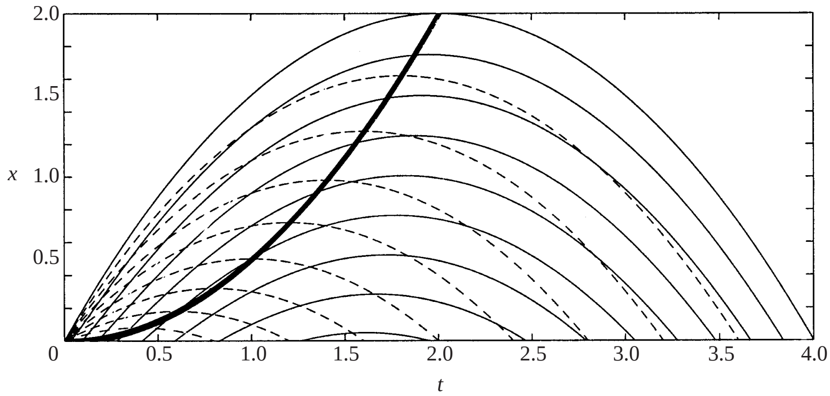

FIGURE 2. Characteristics C_+ (solid line) and C_- (dashed line) and the critical line (heavy line) for the swash solution. The parameters a and b for the characteristics are increased from 0 at the shoreline in steps of 0.2.

on the surface has position $x = x(t)$, $y = h(x, t)$, and since $\mathrm{d}x/\mathrm{d}t = u$, we have

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)^2 h = h_{tt} + 2uh_{xt} + u^2 h_{xx} + (u_t + uu_x)h_x. \tag{3.12}$$

This acceleration has been evaluated and in figure 3 the lines $\mathrm{d}^2y/\mathrm{d}t^2 = 0.1$ and 0.2 are shown. These lines give an indication of where the solution may not be a good representation of the swash; the region where $\mathrm{d}^2y/\mathrm{d}t^2 > 0.1$ is shaded.

4. Overtopping solution

We consider a plane beach cut-off at a point below the maximum run-up, say at $x = E$, $E < 2$. The flow is assumed to fall freely over the edge of the beach. Thus at $x = E$ the flow is initially supercritical, with $u > c$, which implies that there is no influence from the overtopping edge on the flow. This means that the swash solution is unaffected until the flow at the edge slows down and becomes critical at time $t = T_2 = \sqrt{2E}$.

Once the flow velocity drops to critical with Froude number of unity, that is $u = c$, the crest of the beach acts like a weir. The essence of shallow-water theory is that the water depth is negligibly small compared with its horizontal variations, and this also implies that shallow-water time scales are long compared with those related to the depth. Therefore close to the crest the flow may be taken to be almost steady and, as in steady-flow hydraulics, the beach crest 'weir' acts as a 'control point' at which the flow remains critical unless it is submerged. Alternatively, we simply note that once water passes over the crest it is in free fall and can no longer influence flow on the beach: hence the C_- characteristics at the crest of the beach cannot propagate in the $-x$-direction.

The combination of supercritical flow, which needs no boundary condition, and a critical flow boundary condition are sufficient to determine the modification to the swash flow described above by the truncation of the beach. To aid discussion figure 3 shows the relevant portion of the (x, t)-plane divided into four regions.

Region I is the initially dry beach, and is bounded by the moving shoreline (3.1) of the undisturbed swash solution, which reaches $x = E$ at $t = T_1 = 2 - \sqrt{4 - 2E}$.

Region II is the undisturbed swash solution of § 3, whose C_+ and C_- characteristics all originate from the initial singularity. This solution is undisturbed until the flow

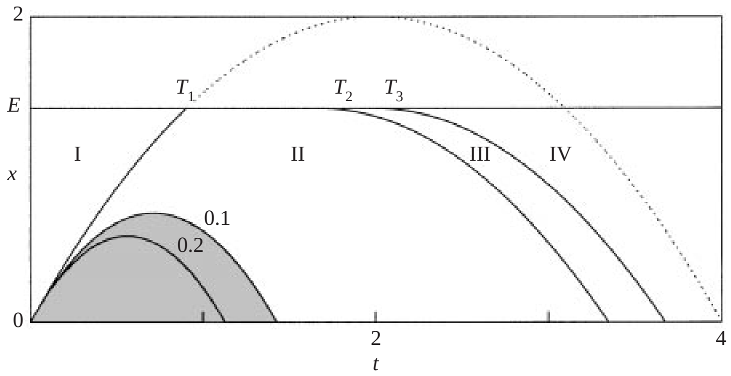

FIGURE 3. General (x, t)-diagram showing different regions of flow. The shaded region is where shallow-water theory is likely to be invalid.

slows sufficiently for a C_- characteristic to travel down the beach from immediately below $x = E$. This characteristic can start, when $u = c$, at time $t = T_2 = \sqrt{2E}$. From equation (3.6) we see that this C_- characteristic bounding region II is

$$x = \sqrt{2E}\, t - \tfrac{1}{2}t^2, \tag{4.1}$$

on which $\beta = \frac{4}{3}\sqrt{2E} - \frac{2}{3}$.

Region III is where the swash is reduced by overtopping, and is discussed further below.

Region IV is where the water has drained away and as indicated in figure 3 starts earlier, at time $t = T_3$, than for the undisturbed swash solution, because water is lost over the edge.

The flow affected by overtopping, region III, can be determined since in that flow all the C_+ characteristics enter from the undisturbed swash flow and hence give

$$\alpha = u + 2c + t = 2. \tag{4.2}$$

If we use this equation to eliminate u from equation (2.7) governing β we find

$$c_t + (2 - 3c - t)c_x = 0. \tag{4.3}$$

At the edge, for $T_2 < t < T_3$, we have both equation (4.2) and $u = c$. Thus we know u, c and β:

$$u = c = \tfrac{1}{3}(2 - T), \quad \beta = \tfrac{2}{3}(2T - 1), \tag{4.4}$$

where we write $t = T$ on $x = E$ since it now is a parameter determining the C_- characteristics. From (4.3) or (2.10) these are

$$x = E - \tfrac{1}{2}(t - T)^2, \tag{4.5}$$

carrying constant values of c and β.

An explicit solution is found by solving (4.5) for

$$T = t - \sqrt{2(E - x)}. \tag{4.6}$$

Then substituting in the expression (4.4) for c, and (4.2) for u, gives

$$c(x, t) = \tfrac{1}{3}[2 - t + \sqrt{2(E - x)}], \tag{4.7}$$

$$u(x, t) = \tfrac{1}{3}[2 - t - 2\sqrt{2(E - x)}], \tag{4.8}$$

and $h = c^2$. The shoreline position is found from $c = 0$ as

$$x_s = E - \tfrac{1}{2}(t-2)^2 \tag{4.9}$$

and thus is similar to the initial swash solution, in falling freely under gravity. The dimensional form of this solution is

$$h^*(x^*, t^*) = \frac{1}{9\cos\gamma}[2\sqrt{A} - \sqrt{g}t^*\sin\gamma + \sqrt{2(EA - x^*\sin\gamma)}]^2, \tag{4.10}$$

$$u^*(x^*, t^*) = \tfrac{1}{3}[2\sqrt{gA} - gt^*\sin\gamma - 2\sqrt{2g(EA - x^*\sin\gamma)}]. \tag{4.11}$$

5. Discussion

The above explicit solution to the nonlinear shallow-water equations was found directly, using the result of Shen & Meyer (1963) for the run-up height near the shoreline to describe a swash event. With the assumption that after a finite time $T_2 = \sqrt{2E}$, the flow becomes critical at the cut-off point $x = E$ we have found the overtopping solution (4.7) and (4.8). It is useful as a test solution for numerical schemes and as a reference solution for studies of wave overtopping. For this latter application we give a few more results and discuss its relevance to practical application.

For overtopping, the greatest interest lies in the flow at $x = E$. At that point the flow is initially supercritical and the swash solution gives

$$u = \frac{2}{3T}(T - T^2 + E), \quad h = \frac{1}{36T^2}(4T - T^2 - 2E)^2, \tag{5.1}$$

giving a flow rate

$$q = uh = \frac{1}{54T^3}(T - T^2 + E)(4T - T^2 - 2E)^2, \tag{5.2}$$

for the time interval $T_1 = 2 - \sqrt{4 - 2E} < T < \sqrt{2E} = T_2$.

Following this, the subcritical flow is

$$u = \tfrac{1}{3}(2 - T), \quad h = \tfrac{1}{9}(2 - T)^2, \tag{5.3}$$

with

$$q = uh = \tfrac{1}{27}(2 - T)^3, \tag{5.4}$$

for the interval $T_2 = \sqrt{2E} < T < 2 = T_3$. It is interesting to note that regardless of the height at which the beach is truncated the overtopping stops at the same time.

From the expressions (5.2) and (5.4) for the volume flux q overtopping the beach, we can find the total volume of overtopping water:

$$\begin{aligned} V(E) = \int_{T_1}^{T_3} q\,\mathrm{d}T &= \tfrac{4}{27}(2\sqrt{2E} - 6E + 3E\sqrt{2E} - E^2) + \tfrac{1}{27}(\sqrt{2} - \sqrt{E})^4 \\ &= \tfrac{1}{27}(4 - 12E + 8E\sqrt{2E} - 3E^2). \end{aligned} \tag{5.5}$$

The function $V(E)$ is plotted in figure 4, distinguishing between the flow during the supercritical and critical stages of overtopping. Since even a small amount of overtopping may be of importance in some circumstances, we also show these quantities on a logarithmic scale in figure 5.

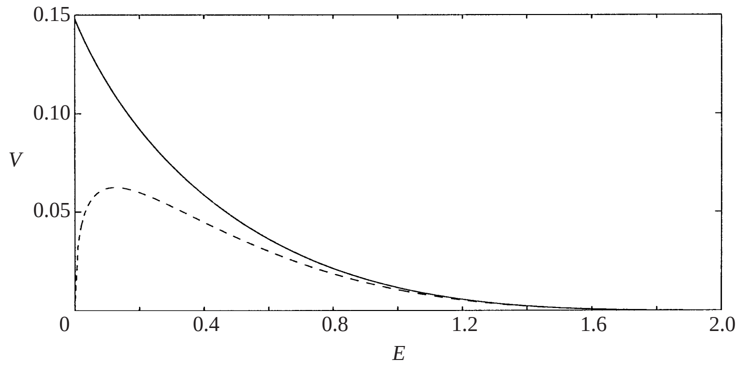

FIGURE 4. Overtopping volume per unit width $V(E)$. Solid line indicates total overtopping. Dashed line indicates overtopping for the supercritical flow only.

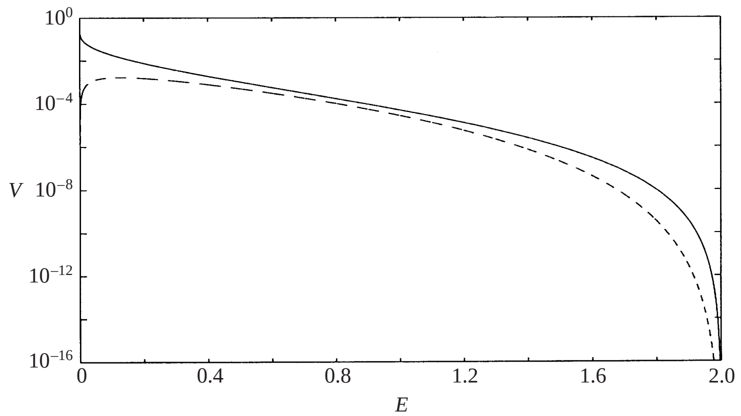

FIGURE 5. $V(E)$ with a logarithmic scale. Solid line is for the total volume per unit width. Dashed line is for the supercritical flow only.

For E near its upper limit, $E = 2 - \delta$, and

$$V \approx \frac{1}{108}\delta^3 + \frac{1}{576}\delta^4 + O(\delta^5). \tag{5.6}$$

In dimensional terms $V^*(E) = 2A^2V(E)/\sin 2\gamma$, per unit length of beach. The variation with beach slope γ is remarkable: for given E, maximum overtopping occurs as the two, inapplicable, limit values $\gamma = 0$ and $\pi/2$ are approached. It appears that for given vertical range of swash, a slope of $\pi/4$ minimizes the overtopping. The limit of $\gamma = 0$ is easy to model since this corresponds to a dam break near the edge of a horizontal bed. The dam break solution (3.7) then applies, since the flow is supercritical at the edge. The flow then continues for all time, unless the initial volume of water is bounded.

Unfortunately real life waves are not as simple as the above analysis. For a start, we have chosen a special solution for the swash. It is clear that the value of α on the C_+ characteristics 'feeding' the swash on $x = 0$ would in general deviate from the initial value of 2. Also real fluid effects such as friction with the bed influence the flow, though indications from comparisons between computations with the shallow-water equations and laboratory experiments (Packwood 1980; Barnes 1996) indicate that frictional effects in the uprush are small, unless γ is as small as 0.02 or water depth is less than 2 mm.

Regardless of the special nature of this overtopping solution it can provide a useful reference against which computational and experimental results for swash overtopping may be compared. It should be borne in mind that the swash solution of § 3 is only relevant when the shoreline is moved impulsively due to arrival of a bore or breaking wave. On steep slopes non-breaking waves are also a frequent cause of swash.

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